

Darboux transformation and analytic solutions of the discrete \mathcal{PT} -symmetric nonlocal nonlinear Schrödinger equation

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Abstract

In this letter, for the discrete parity-time-symmetric nonlocal nonlinear Schrödinger equation, we construct the Darboux transformation, which provides an algebraic iterative algorithm to obtain a series of analytic solutions from a known one. To illustrate, the breathing-soliton solutions, periodic-wave solutions and localized rational soliton solutions are derived with the zero and plane-wave solutions as the seeds. The properties of those solutions are also discussed, and particularly the asymptotic analysis reveals all possible cases of the interaction between the discrete rational dark and antidark solitons.

Keywords: Nonlocal nonlinear Schrödinger equation; Soliton solutions; Darboux transformation; Parity-time symmetry

1 Introduction

In 1998, Bender and Boettcher first pointed out that a non-Hermitian Hamiltonian can have the real and positive eigenvalues provided that it meets the combined parity and time reversal symmetry (usually called the \mathcal{PT} symmetry) [1]. Since then, the non-Hermitian but \mathcal{PT} -symmetric systems have appeared in many areas such as nonlinear optics [2], complex crystal [3], quantum chromodynamics [4], Bose-Einstein condensates [5] in addition to quantum mechanics [1]. Also, the notion of \mathcal{PT} symmetry has been extended to nonlinear integrable systems in mathematical physics [6–14]. It should be noted that exactly solvable integrable systems are ubiquitous in nonlinear science and play an important role in describing various nonlinear wave phenomena, like solitary waves [15] and rogue waves [16].

In 2013, Ablowitz and Musslimani proposed the following continuous \mathcal{PT} -symmetric nonlocal nonlinear Schrödinger (NNLS) equation [6]:

$$i q_t(x, t) = q_{xx}(x, t) \pm 2 q(x, t) q^*(-x, t) q(x, t), \quad (1)$$

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where $*$ denotes complex conjugation, the nonlinear term is nonlocally dependent on the values of q both at x and $-x$, and the self-induced potential $V(x, t) = \pm 2q(x, t)q^*(-x, t)$ obey the \mathcal{PT} -symmetric condition $V(x, t) = V^*(-x, t)$. Eq. (1) is integrable in the sense that it admits the Lax pair and an infinite number of conservation laws, and thus its initial-value problem can be solved by the inverse scattering transform (IST) [6]. Recently, much effort has been made to construct the exact analytic solutions of Eq. (1), including the singular pure soliton solution on the vanishing background [6], static bright and dark soliton solutions [7, 8], periodic and hyperbolic soliton solutions [8], exponential and rational soliton solutions on the continuous wave background [9, 10], and Peregrine-type rogue waves on the finite background [11]. In addition, the integrable vector extension and two-dimensional generalization of Eq. (1) have also been discussed [12, 13].

In this letter, we will study the discrete \mathcal{PT} -symmetric NNLS equation [14]:

$$i \frac{dQ_n}{dt} = Q_{n+1} - 2Q_n + Q_{n-1} - \epsilon Q_n Q_{-n}^* (Q_{n+1} + Q_{n-1}), \quad (2)$$

which is also an integrable Hamiltonian model [14], where $Q_n(t)$ is a complex-valued function, n is an integer, $\epsilon = \mp 1$ denotes the focusing and defocusing cases, respectively. The IST scheme of solving Eq. (2) has been established [14], and some singular soliton solutions and periodic-wave solutions of Eq. (2) with $\epsilon = -1$ have also been obtained by the Hirota method [17]. The main work in this study is to construct the N -time iterated Darboux transformation (DT), which provides an algebraic iterative algorithm to obtain the analytic solutions of Eq. (2) from a known one. To illustrate, with the zero and plane-wave solutions as the seeds, we derive the breathing-soliton solutions, periodic-wave solutions and localized rational soliton solutions. In particular, via asymptotic analysis we reveal three types of elastic interactions between rational dark (RD) and rational antidark (RAD) solitons, and two types of degenerate two-soliton interactions.

2 Darboux transformation of Eq. (2)

The Lax pair of Eq. (2) takes the form

$$\Phi_{n+1} = U_n \Phi_n, \quad U_n = \begin{pmatrix} z & Q_n \\ R_n & z^{-1} \end{pmatrix}, \quad (3a)$$

$$\frac{d\Phi_n}{dt} = V_n \Phi_n, \quad V_n = \begin{pmatrix} i R_{n-1} Q_n + i(1 - z^2) & -i z Q_n + i z^{-1} Q_{n-1} \\ -i z R_n + i z^{-1} R_{n-1} & -i Q_{n-1} R_n - i(1 - z^{-2}) \end{pmatrix}, \quad (3b)$$

with $R_n = \epsilon Q_{-n}^*$, where $\Phi_n = (\phi_{1,n}, \phi_{2,n})^T$ is the vector eigenfunction, z is a complex spectral parameter. As a special gauge transformation, the DT comprises of the eigenfunction and potential transformations. For the once-iterated DT, we take the eigenfunction transformation on System (3) be of the form

$$\Phi_n^{[1]} = T_n^{[1]} \Phi_n, \quad T_n^{[1]} = \begin{pmatrix} m_1(t) (z - a_n(t) z^{-1}) & -b_n(t) \\ -c_n(t) & h_1(t) (z^{-1} - d_n(t) z) \end{pmatrix}, \quad (4)$$

where $T_n^{[1]}$ is the once Darboux matrix, $m_1(t)$, $h_1(t)$, $a_n(t)$, $b_n(t)$, $c_n(t)$ and $d_n(t)$ are to be determined, and $\Phi_n^{[1]} = (\phi_{1,n}^{[1]}, \phi_{2,n}^{[1]})^T$ is the once iterated eigenfunction.

From the knowledge of DT, $\Phi_n^{[1]}$ is required to satisfy

$$T_{n+1}^{[1]} U_n = U_n^{[1]} T_n^{[1]}, \quad \frac{d}{dt} T_n^{[1]} + T_n^{[1]} V_n = V_n^{[1]} T_n^{[1]}, \quad (5)$$

where $U_n^{[1]}$ and $V_n^{[1]}$ are the same as U_n and V_n except that Q_n and Q_{-n}^* are replaced by $Q_n^{[1]}$ and $Q_{-n}^{*[1]}$, respectively. We note the fact [14] that if $\Phi_{1,n} = (f_{1,n}, g_{1,n})^T$ satisfies System (3) with $z = z_1$, then $\bar{\Phi}_{1,n} = w_{-n}^* (g_{1,1-n}^*, -\epsilon f_{1,1-n}^*)^T$ is also a solution of System (3) with $z = z_1^*$, where $w_n = \prod_{k=-\infty}^n \frac{1}{1-\epsilon Q_k Q_{-k}^*}$. Thus, a_n, b_n, c_n and d_n can be determined by demanding $T_n^{[1]}|_{z=z_1} \Phi_{1,n} = \mathbf{0}$ and $T_n^{[1]}|_{z=z_1^*} \bar{\Phi}_{1,n} = \mathbf{0}$. Furthermore, one can check that Eq. (5) is satisfied if $Q_n^{[1]}$ and $Q_{-n}^{*[1]}$ are given by

$$Q_n^{[1]} = -\frac{m_1 Q_n + b_n}{h_1 d_n}, \quad R_n^{[1]} = -\frac{c_n + h_1 R_n}{m_1 a_n}, \quad (6)$$

where the symmetry reduction $R_n^{[1]} = \epsilon Q_{-n}^{*[1]}$ holds if taking $h_1(t) = |z_1|$ and $m_1(t) = |z_1|^{-1}$.

Next, we construct the N -time iterated DT of Eq. (2). In doing so, we successively implement the once-iterated eigenfunction transformation, that is,

$$\Phi_n^{[N]} = T_{N,n}^{[1]} \cdots T_{1,n}^{[1]} \Phi_n, \quad T_{k,n}^{[1]} = \begin{pmatrix} m_k(t) (z - a_{k,n}(t) z^{-1}) & -b_{k,n}(t) \\ -c_{k,n}(t) & h_k(t) (z^{-1} - d_{k,n}(t) z) \end{pmatrix}, \quad (7)$$

where $1 \leq k \leq N$, $\Phi_n^{[N]} = (f_{1,n}^{[N]}, g_{1,n}^{[N]})^T$ is the N -time iterated eigenfunction, $h_k(t) = |z_k|$ and $m_k(t) = |z_k|^{-1}$. The N -time iterated Darboux matrix $T_n^{[N]}$ can be written as

$$T_n^{[N]} = T_{N,n}^{[1]} \cdots T_{1,n}^{[1]} = \begin{pmatrix} A_N(n, t, z) & B_N(n, t, z) \\ C_N(n, t, z) & D_N(n, t, z) \end{pmatrix}, \quad (8)$$

with $A_N = \mathfrak{M} z^N + \sum_{j=1}^N a_n^{(j)}(t) z^{N-2j}$, $B_N = \sum_{j=1}^N b_n^{(j)}(t) z^{N+1-2j}$, $C_N = \sum_{j=1}^N c_n^{(j)}(t) z^{N+1-2j}$, $D_N = \mathfrak{M}^{-1} z^{-N} + \sum_{j=1}^N d_n^{(j)}(t) z^{N+2-2j}$ and $\mathfrak{M} = \prod_{k=1}^N |z_k|^{-1}$.

Similarly, $a_n^{(j)}(t)$, $b_n^{(j)}(t)$, $c_n^{(j)}(t)$ and $d_n^{(j)}(t)$ ($1 \leq j \leq N$) can be uniquely determined by requiring $T_n^{[N]}|_{z=z_k} \Phi_{k,n} = \mathbf{0}$ and $T_n^{[N]}|_{z=z_k^*} \bar{\Phi}_{k,n} = \mathbf{0}$ ($1 \leq k \leq N$), where $\Phi_{k,n} = (f_{k,n}, g_{k,n})^T$ and $\bar{\Phi}_{k,n} = w_{-n}^* (g_{k,1-n}^*, -\epsilon f_{k,1-n}^*)^T$ ($w_n = \prod_{k=-\infty}^n \frac{1}{1-\epsilon Q_k Q_{-k}^*}$) are, respectively, the solutions of System (3) with $z = z_k$ and $z = z_k^*$. Via Cramer's rule, the N -time iterated potentials $Q_n^{[N]}$ and $R_n^{[N]}$ can be represented as

$$Q_n^{[N]} = \frac{Q_n \mathfrak{M} - b_n^{(1)}}{d_n^{(1)}}, \quad R_n^{[N]} = \frac{R_n \mathfrak{M}^{-1} - c_n^{(N)}}{a_n^{(N)}}, \quad (9)$$

with

$$a_n^{(N)} = (-1)^N \mathfrak{M} \frac{\tau_n(N, N-2; N-1, N-1)}{\tau_n(N-2, N; N-1, N-1)}, \quad b_n^{(1)} = (-1)^{N+1} \mathfrak{M} \frac{\tau_n(N, N; N-3, N-1)}{\tau_n(N-2, N; N-1, N-1)},$$

$$c_n^{(N)} = (-1)^{N+1} \mathfrak{M}^{-1} \frac{\tau_n(N-1, N-3; N, N)}{\tau_n(N-1, N-1; N, N-2)}, \quad d_n^{(1)} = (-1)^N \mathfrak{M}^{-1} \frac{\tau_n(N-1, N-1; N-2, N)}{\tau_n(N-1, N-1; N, N-2)},$$

where the determinant $\tau_n(M, L; M', L')$ is defined as

$$\tau_n(M, L; M', L') = \begin{vmatrix} F_n(M, L) & G_n(M', L') \\ G_{1-n}^*(M, L) & -\epsilon F_{1-n}^*(M', L') \end{vmatrix}, \quad d_n^{(1)} = a_n^{(N)} \mathfrak{M}^4, \quad (10)$$

$$\text{with } F_n(M, L) = (z_k^{M+2-2j} f_{k,n})_{\substack{1 \leq k \leq N, \\ 1 \leq j \leq \frac{M+L}{2}+1}}, \quad G_n(M', L') = (z_k^{M'+2-2j} g_{k,n})_{\substack{1 \leq k \leq N, \\ 1 \leq j \leq \frac{M'+L'}{2}+1}}, \quad F_{1-n}^*(M', L') = (z_k^{*M'+2-2j} f_{k,1-n}^*)_{\substack{1 \leq k \leq N, \\ 1 \leq j \leq \frac{M'+L'}{2}+1}}, \quad G_{1-n}^*(M, L) = (z_k^{*M+2-2j} g_{k,1-n}^*)_{\substack{1 \leq k \leq N, \\ 1 \leq j \leq \frac{M+L}{2}+1}}. \quad \text{Finally, it can be proved}$$

that the N -time iterated potentials in Eq. (9) obey the reduction relation $R_n^{[N]} = \epsilon Q_{-n}^{*[N]}$ (see Appendix A). Hence, we can safely say that Transformations (7) and (9) constitute the N -time iterated DT of Eq. (2).

3 Analytic solutions on the vanishing and plane-wave backgrounds

In this section, based on the above-obtained DT algorithm, we will choose the zero solution and plane-wave solutions as the seeds to derive some new analytic solutions of Eq. (2), including the breathing-soliton solutions, periodic-wave solutions and localized rational soliton solutions.

A. Breathing-soliton solutions on the vanishing background

First, starting from the seed $Q_n = 0$, we solve System (3) with $Q_n = 0$ and $z = z_1$, yielding

$$(f_{1,n}, g_{1,n}) = (\alpha_1 z_1^n e^{it(1-z_1^2)}, \beta_1 z_1^{-n} e^{-it(1-z_1^{-2})}), \quad (11)$$

where α_1 and β_1 are two nonzero complex constants. Substituting (11) into Eq. (6) gives rise to

$$Q_n^{[1]} = \frac{\gamma_1^* (z_1^*)^{2n-1} (z_1^{*2} - z_1^2) e^{-i\chi_1^{*2} t}}{\epsilon |z_1|^2 + |\gamma_1|^2 \left(\frac{z_1^*}{z_1}\right)^{2n} e^{i(\chi_1^2 - \chi_1^{*2}) t}}, \quad (\chi_1 = z_1 - z_1^{-1}, \gamma_1 = \frac{\beta_1}{\alpha_1}), \quad (12)$$

which has also been obtained by the IST method [14] and by the Hirota method [17]. Note that this solution is singular in the continuous limit. However, because n is an integer variable, all the singularities can be avoided if taking $\arg(z_1) \neq \frac{m\pi}{4n}$ ($m, n \in \mathbb{Z}$) for $|z_1| = 1$ and $|\gamma_1| \neq 1$ for $|z_1| \neq 1$. In illustration, Fig. 1 presents a discrete breathing soliton which oscillates periodically along the n axis.

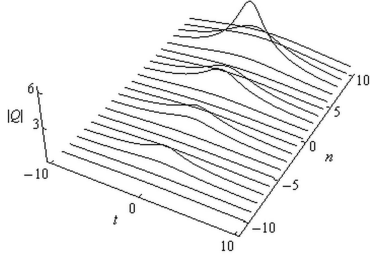


Figure 1: Breathing one-soliton solution with $\epsilon = 1$ and $z_1 = 1 + \frac{i}{3}$.

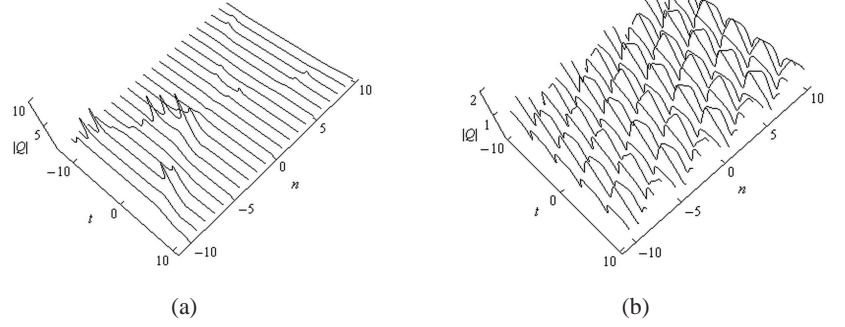


Figure 2: (a) Breathing-soliton and (b) periodic-wave solutions on the plane-wave background, where $c = \frac{1}{2}$, $\epsilon = 1$, $\alpha_1 = 1$, (a) $\beta_1 = 1 + 2i$, $z_1 = -\frac{3}{5} - \frac{i}{5}$, and (b) $\beta_1 = 1$, $z_1 = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}i}{2}$.

B. Breathing-soliton and periodic-wave solutions on the plane-wave background

Second, we try to construct the soliton solutions on the nonvanishing background. It is easy to find that Eq. (2) admits the plane-wave solution $Q_n = c e^{2i\epsilon|c|^2 t}$, where c is a complex constant. Then, we substitute the solution into System (3) and solve the resulting equations with $z = z_1$. If c and z_1 satisfy the condition $\chi_1^2 + 4\epsilon|c|^2 \neq 0$ ($\chi_1 = z_1 - z_1^{-1}$), one can obtain

$$\begin{pmatrix} f_{1,n} \\ g_{1,n} \end{pmatrix} = \begin{pmatrix} e^{i\epsilon|c|^2 t} (\alpha_1 e^{n \ln \mu_1^- - i\chi_1 \mu_1^- t} + \beta_1 e^{n \ln \mu_1^+ - i\chi_1 \mu_1^+ t}) \\ \frac{e^{-i\epsilon|c|^2 t}}{c} [(z_1 - \mu_1^-) \alpha_1 e^{n \ln \mu_1^- - i\chi_1 \mu_1^- t} + (\mu_1^+ - z_1) \beta_1 e^{n \ln \mu_1^+ - i\chi_1 \mu_1^+ t}] \end{pmatrix}, \quad (13)$$

where $\mu_1^\pm = \frac{z_1^2 + 1 \pm \sqrt{(z_1^2 - 1)^2 + 4\epsilon|c|^2 z_1^2}}{2z_1}$, and α_1 and β_1 are two nonzero complex parameters. Then, with substitution of (13) into Eq. (6), we have the breathing-soliton solution (which is omitted here because its expression is too long and complicated) on the plane-wave background. With suitable choice of the involved parameters, the solution has no singularity for all integers n . But different from the exponential dark and antidark solitons in the defocusing case of Eq. (1) [9], there appear two breathing solitons under the nonsingular condition and they exhibit the elastic interactions on the background [see Fig. 2(a)]. In particular, with $|z_1| = 1$ the solution reduces to the triangular periodic solution and describes the interaction of periodic waves [see Fig. 2(b)].

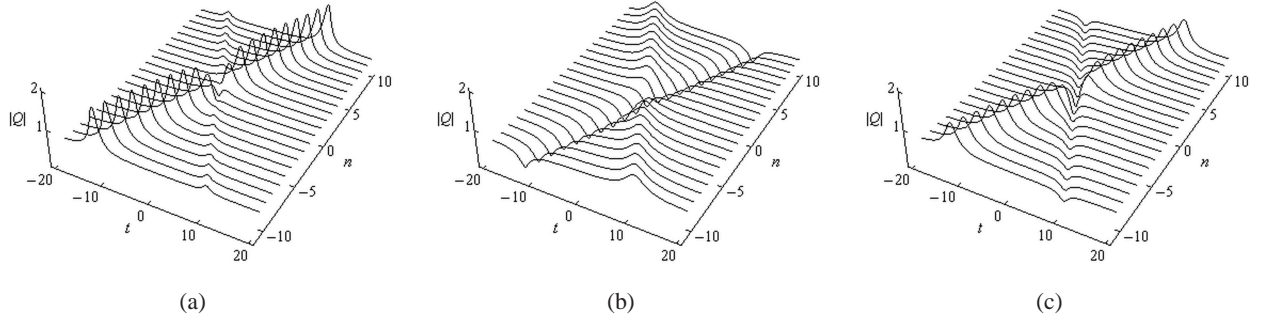


Figure 3: Three types of elastic two-soliton interactions with $c = \frac{1}{2}$ and $\theta = \frac{\pi}{6}$, where (a) RAD-RAD interaction ($\gamma_1 = 1 + 1.6i$), (b) RAD-RD interaction ($\gamma_1 = 1 - i$), and (c) RD-RAD interaction ($\gamma_1 = 1 + 2i$).

C. Localized rational soliton solutions on the plane-wave background

Third, we also choose the seed solution $Q_n = c e^{2i\epsilon|c|^2 t}$ but consider the particular case $\chi_1^2 + 4\epsilon|c|^2 = 0$. Note that if $\epsilon = -1$ or $|c| \geq 1$, z_1 reduces to a pure real or imaginary number, so that the DT is trivial and no new solution can be generated. Thus, we must impose $\epsilon = 1$ and $|c| < 1$, and obtain that $z_1 = e^{i\theta}$ ($\tan \theta = \frac{\pm|c|}{\sqrt{1-|c|^2}}$). In this case, System (3) has the following solution:

$$\begin{pmatrix} f_{1,n} \\ g_{1,n} \end{pmatrix} = \begin{pmatrix} \frac{i \sin \theta}{c^*} e^{i|c|^2 t} \cos^n \theta \cdot e^{t \sin(2\theta)} [n + t \sin(2\theta) + \gamma_1 - i \cot \theta] \\ e^{-i|c|^2 t} \cos^n \theta \cdot e^{t \sin(2\theta)} [n + t \sin(2\theta) + \gamma_1] \end{pmatrix}, \quad (14)$$

where γ_1 is a nonzero complex constant. Substituting Eq. (14) into Eq. (6), we have

$$Q_n^{[1]} = c e^{2i|c|^2 t} \frac{(\xi - K - i \cot \theta)(\eta + K^* + i \cot \theta) - \frac{\csc^2 \theta}{4}}{\xi \eta - K \eta + K^* \xi - |K|^2 - \frac{\csc^2 \theta}{4}}, \quad (15)$$

with $\xi = n + \sin(2\theta) \cdot t$, $\eta = n - \sin(2\theta) \cdot t$, $K = \frac{i \cot \theta - 1}{2} - \gamma_1$. It can be proved that Solution (15) has no singularity if and only if $\text{Im}(\gamma_1) \neq \frac{\cot \theta}{2}$.

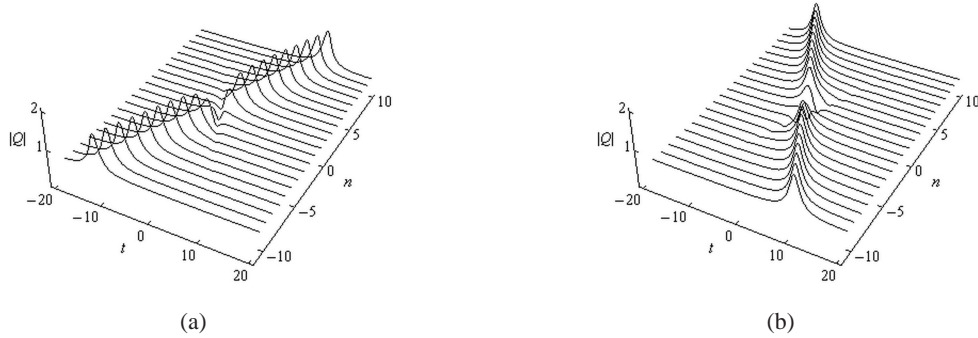


Figure 4: Degenerate two-soliton interactions with $c = \frac{1}{2}$, $\theta = \frac{\pi}{6}$, (a) $\gamma_1 = 1 + \sqrt{3}i$ and (b) $\gamma_1 = 1$.

When $|t| \rightarrow \infty$, we obtain two asymptotic expressions of Solution (15) as follows:

$$Q_n^{[1]} \rightarrow \begin{cases} S_n^{(I)} = c e^{2i|c|^2 t} (1 + \frac{i \cot \theta}{K - \xi}), & n + \sin(2\theta) \cdot t \sim 0, \\ S_n^{(II)} = c e^{2i|c|^2 t} (1 + \frac{i \cot \theta}{K^* + \eta}), & n - \sin(2\theta) \cdot t \sim 0, \end{cases} \quad (16)$$

where $S_n^{(I)}$ describes an RD or RAD soliton for $1 - \text{Im}(\gamma_1) \tan \theta < 0$ or $1 - \text{Im}(\gamma_1) \tan \theta > 0$, while $S_n^{(II)}$ represents an RD or RAD soliton for $\text{Im}(\gamma_1) \cot \theta < 0$ or $\text{Im}(\gamma_1) \cot \theta > 0$. Similar to the defocusing case occurring in Eq. (1) [10], one can obtain the elastic interactions of the discrete RAD-RAD, RAD-AD and AD-RAD soliton pair on a plane-wave background, and there is no phase shift for the interacting solitons [see Figs. 3(a)–3(c)]. Particularly when $1 - \text{Im}(\gamma_1) \tan \theta = 0$ or $\text{Im}(\gamma_1) \cot \theta = 0$, the asymptotic soliton $S_n^{(I)}$ or $S_n^{(II)}$ disappears for large values of t , as shown in Figs. 4(a) and 4(b). The relevant parametric conditions for five different asymptotic patterns of Solution (15) are presented in Table 1.

Table 1: Asymptotic patterns of Solution (15) under different parametric conditions.

Parametric conditions	Asymptotic soliton $S_n^{(I)}$	Asymptotic soliton $S_n^{(II)}$
$1 - \text{Im}(\gamma_1) \tan \theta > 0, \text{Im}(\gamma_1) \cot \theta > 0$	RAD soliton	RAD soliton
$1 - \text{Im}(\gamma_1) \tan \theta > 0, \text{Im}(\gamma_1) \cot \theta < 0$	RAD soliton	RD soliton
$1 - \text{Im}(\gamma_1) \tan \theta < 0, \text{Im}(\gamma_1) \cot \theta > 0$	RD soliton	RAD soliton
$1 - \text{Im}(\gamma_1) \tan \theta = 0$	Vanish	RAD soliton
$\text{Im}(\gamma_1) \cot \theta = 0$	RAD soliton	Vanish

4 Concluding remarks

Currently, it has been an important concern to study the \mathcal{PT} -symmetric integrable systems in nonlinear mathematical physics. In this letter, for the discrete \mathcal{PT} -symmetric NNLS equation (2), we have constructed its N -time iterated DT and have represented the iterated solutions in terms of some simple determinant. To illustrate, with the zero and plane-wave solutions as the seeds, we have derived the breathing-soliton solutions, periodic-wave solutions and localized rational soliton solutions. Also, we have discussed the properties of those solutions, and particularly revealed the elastic interactions of the discrete RAD-RAD, RAD-AD and AD-RAD soliton pair on the plane-wave background. It should be mentioned that the defocusing Ablowitz-Ladik model admits the exploding rogue-wave solutions which develop singularity at a certain specific time [18]. Very differently, the rational solution (15) has no singularity only if $\text{Im}(\gamma_1) \neq \frac{\cot \theta}{2}$, and can exhibit the elastic interactions between two traveling rational solitons. In the future, it is worth to further study the stability of localized rational soliton solutions and the dynamical properties of the multi-iterated solutions via the DT.

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Appendix A: Proof of the reduction relation $R_n^{[N]} = \epsilon Q_{-n}^{*[N]}$ in Eq. (9)

Recall that $\Phi_{k,n} = (f_{k,n}, g_{k,n})^T$ and $\bar{\Phi}_{k,n} = w_{-n}^*(g_{k,1-n}^*, -\epsilon f_{k,1-n}^*)^T$ satisfy System (3) with $z = z_k$ and $z = z_k^*$, respectively. Thus, we can arrive at the following identity relations:

$$\tau_n(N-1, N-1; N, N-2) = \mathfrak{M}^{-2} \tau_n(N-2, N; N-1, N-1), \quad (\text{A.1})$$

$$F_n(N-k, N-l) + Q_n G_n(N-k-1, N-l+1) = F_{n+1}(N-k, N-l), \quad (\text{A.2})$$

$$\epsilon Q_{-n}^* F_n(N-k, N-l) + G_n(N-k-1, N-l+1) = G_{n+1}(N-k, N-l), \quad (\text{A.3})$$

$$\begin{aligned} G_{1-n}^*(N-k, N-l) - \epsilon Q_n F_{1-n}^*(N-k-1, N-l+1) \\ = (1 - \epsilon Q_n Q_{-n}^*) G_{-n}^*(N-k-1, N-l+1), \end{aligned} \quad (\text{A.4})$$

$$\begin{aligned} Q_{-n}^* G_{1-n}^*(N-k, N-l) - F_{1-n}^*(N-k-1, N-l+1), \\ = (\epsilon Q_n Q_{-n}^* - 1) F_{-n}^*(N-k, N-l). \end{aligned} \quad (\text{A.5})$$

By virtue of the above identities, $Q_n^{[N]}$ and $R_n^{[N]}$ in Eq. (9) can be simplified as

$$\begin{aligned} Q_n^{[N]} &= \frac{Q_n \tau_n(N-2, N; N-1, N-1) + (-1)^N \tau_n(N, N; N-3, N-1)}{(-1)^N \tau_n(N-1, N-1; N-2, N)} \\ &= \frac{\begin{vmatrix} F_{n+1}(N-1, N-1) & F_n(-N, N) & G_n(N-3, N-1) \\ (1 - \epsilon Q_n Q_{-n}^*) G_{-n}^*(N-1, N-1) & G_{1-n}^*(-N, N) & -\epsilon F_{1-n}^*(N-3, N-1) \end{vmatrix}}{\begin{vmatrix} F_{n+1}(N-2, N) & G_n(N-2, N) \\ (1 - \epsilon Q_n Q_{-n}^*) G_{-n}^*(N-2, N) & -\epsilon F_{1-n}^*(N-2, N) \end{vmatrix}}, \end{aligned} \quad (\text{A.6})$$

$$\begin{aligned} R_n^{[N]} &= \frac{\epsilon Q_{-n}^* \tau_n(N-1, N-1; N, N-2) + (-1)^N \tau_n(N-1, N-3; N, N)}{(-1)^N \tau_n(N, N-2; N-1, N-1)} \\ &= \frac{\begin{vmatrix} F_{n+1}(N-2, N-2) & G_{n+1}(1-N, N-1) & G_n(N, N-2) \\ (1 - \epsilon Q_n Q_{-n}^*) G_{-n}^*(N-2, N-2) & -\epsilon F_{-n}^*(1-N, N-1) & -\epsilon F_{1-n}^*(N, N-2) \end{vmatrix}}{(-1)^N \begin{vmatrix} F_{n+1}(N-1, N-1) & G_n(N-1, N-1) \\ (1 - \epsilon Q_n Q_{-n}^*) G_{-n}^*(N-1, N-1) & -\epsilon F_{1-n}^*(N-1, N-1) \end{vmatrix}}. \end{aligned} \quad (\text{A.7})$$

Taking complex conjugate and changing $n \rightarrow -n$ for $Q_n^{[N]}$ in Eq. (A.6), one can check that $R_n^{[N]} = \epsilon Q_{-n}^{*[N]}$ is exactly satisfied.